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Action diffusion for symplectic maps with a noisy linear frequency

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Abstract. We consider an area preserving map in the neighbourhood of an elliptic fixed point, whose linear frequency is stochastically perturbed. The nonlinearity couples the random motion in the phase with the action which exhibits a diffusive behaviour. If the unperturbed dynamics is almost integrable and no macroscopic resonant structures are present in the phase space, a Fokker–Planck equation for the action diffusion is derived and its solution shows an excellent agreement with the simulation of the process. The key points are the description of the unperturbed motion by using the normal forms and the derivation of a stochastically perturbed interpolating Hamiltonian for which the action diffusion coefficient is analytically computed. The angle averaging is justified by the much faster time scale on which the angle relaxes to a uniform distribution.

1. Introduction

The action diffusion in almost integrable Hamiltonian systems (Arnold's diffusion) has been intensively investigated [1–3], but both the analytical results and the numerical experiments show that its physical relevance is limited by the critical dependence on the initial conditions. As a consequence the slow diffusion observed in some physical experiments may be explained by the presence of a small stochastic noise which is unavoidable in realistic situations.

The effect of a small amount of noise on area-preserving maps in the stochastic regime has already been investigated [4, 5]: the noise is inserted to avoid the singular behaviour of orbits with a long correlation time, which are present in a strongly perturbed map and the techniques used do not work when the amplitude of the noise tends to zero.

Here we analyse the effect of small noise in the almost integrable case: by introducing a diffusion time, we justify a Fokker–Planck equation for the distribution function in the unperturbed action in the limit of small amplitude for the noise. We consider a polynomial area-preserving map in the neighbourhood of an elliptic fixed point at the origin whose linear frequency is stochastically perturbed: the unperturbed phase space of the map shows a region of bounded motion almost completely foliated with invariant curves and a region $\mathcal{A}$ of unbounded motion. The small stochastic perturbation determines a random motion across the invariant curves and a diffusion in the action variable up to the boundary of $\mathcal{A}$, whereas the angle relaxes very rapidly to a uniform distribution. As a consequence we can consider the angle variable as a fast variable and an averaging procedure is justified by stochastic theory of adiabatic invariance [6].

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The results we obtain are relevant for beam dynamics, since the transverse motion of a particle is described by a symplectic map; the presence of ripples in the currents feeding the quadrupoles determines a perturbation of the linear frequency, which can be considered as a stochastic process. The experiments on the SPS at CERN [7] show that the loss of the beam is due to the combined effect of the nonlinear magnetic fields and the ripples. For this reason we have supposed that the noise affects mainly the linear frequencies (tunes); indeed in this case no diffusion occurs if the nonlinearity is switched off.

In this paper we describe a procedure which allows one to obtain a Fokker–Planck equation for the action variable computed by means of the Birkhoff normal forms [8, 9]; the key point is to construct an interpolating Hamiltonian for the stochastic map up to second order in the perturbation parameter (i.e. the strength of the noise) in order to write the Liouville equation for the distribution function. Then we use the approach developed by Gurievich *et al* [10] to obtain a Fokker–Planck equation in the action-angle variables: since the angle has a fast relaxation to a uniform distribution, by an averaging method we drop the angle dependence and obtain a diffusion equation for the distribution function in the action [11]. The problem can be solved in a systematic way for any map known in analytic form or by a truncated Taylor expansion, as it is the case in accelerator physics. The computation of the diffusion coefficient directly from the map could be achieved if one justifies a central limit theorem for the random walk in the action variable. This approach, which needs the results of the theory of the stochastic processes, will be considered in the future.

We explicitly study the Hénon quadratic map and we compare the solution of the Fokker–Planck equation with the distribution function numerically computed by iterating the map. When no low-order resonances are present in the stable region, we find an excellent agreement between the simulations and the analytical results. The technique proposed here can be extended to symplectic maps of higher dimensionality.

The plan of the paper is the following: in section 2 we briefly review the diffusion problem for polynomial maps and formulate the model of a stochastically perturbed map. In section 3 we compute the interpolating Hamiltonian and determine the diffusion coefficient. In section 4 some numerical examples are discussed.

2. Stochastic maps

We consider a stochastic area-preserving map defined according to

$$F_\epsilon : \begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = R(\epsilon\xi_n) F(x_n, p_n) \qquad F(x, p) = R(\omega) \begin{pmatrix} x \\ p + b(x) \end{pmatrix}. \tag{2.1}$$

In order to define approximate action-angle variables we introduce an interpolating Hamiltonian whose phase flow at integer times agrees with the iteration of the map (2.1). It is convenient to use complex coordinates $z = x - \mathrm{i}p$ so that the map reads

$$z_{n+1} = \mathrm{e}^{\mathrm{i}\epsilon\xi_n} F(z_n, z_n^*). \tag{2.2}$$

Assuming that the linear frequency $\omega/2\pi$ is not a rational number, we compute the normalizing transformation $z = \Phi(\zeta, \zeta^*)$ and the normal form U, by solving the conjugation equation

$$F \circ \Phi = \Phi \circ (U + E_N) \tag{2.3}$$

where $U_N = \zeta \mathrm{e}^{\mathrm{i}\Omega(\zeta\zeta^*)}$ and $E_N = \mathrm{O}(|\zeta|^{N+1})$ is the error term. If we introduce the action-angle variables (J, Θ)

$$\zeta = \sqrt{2J}\mathrm{e}^{\mathrm{i}\Theta} \tag{2.4}$$

the normal map U reads

$$\begin{aligned} J_{n+1} &= J_n \\ \Theta_{n+1} &= \Theta_n + \Omega(J_n) \end{aligned} \tag{2.5}$$

and agrees with the phase flow at integer times of the Hamiltonian

$$H_0(J) = \omega J + h_2 J^2 + \cdots + h_{[(N+1)/2]} J^{[(N+1)/2]} \tag{2.6}$$

where H_0 is defined by $\mathrm{d}H_0/\mathrm{d}J = \Omega(J)$. By introducing the Lie operator D_H associated with the Hamiltonian H,

$$D_H f(J,\Theta) = [f, H] = \frac{\partial f}{\partial \Theta}\frac{\partial H}{\partial J} - \frac{\partial f}{\partial J}\frac{\partial H}{\partial \Theta} \tag{2.7}$$

where [,] denotes the usual Poisson bracket, the initial map F_ϵ can be written in the form

$$M_\epsilon \begin{pmatrix} J_n \\ \Theta_n \end{pmatrix} = \exp(\epsilon \xi_n D_{V(J,\Theta)}) \circ \exp(D_{H\circ(J)}) \begin{pmatrix} J_n \\ \Theta_n \end{pmatrix} \tag{2.8}$$

up to an error of order $\mathrm{O}(J^{[N+1/2]})$; the perturbation $V(J,\Theta)$ is given by

$$V(J,\Theta) = \tfrac{1}{2}\Phi(\sqrt{2J}\mathrm{e}^{\mathrm{i}\Theta}, \sqrt{2J}\mathrm{e}^{-\mathrm{i}\Theta})\Phi^*(\sqrt{2J}\mathrm{e}^{\mathrm{i}\Theta}, \sqrt{2J}\mathrm{e}^{-\mathrm{i}\Theta}). \tag{2.9}$$

3. The interpolating Hamiltonian

In order to determine an interpolating Hamiltonian for the map (2.8), we introduce the stochastic process $\Xi(t)$ defined according to $\Xi(t) = \xi_n$ when $t \in [n, n+1[$ and we look for an interpolating Hamiltonian of the form

$$H(J,\Theta,\epsilon\Xi(t)) = H_0(J) + \epsilon\Xi(t)H_1(J,\Theta) + \epsilon^2\Xi^2(t)H_2(J,\Theta) + \mathrm{O}(\epsilon^3). \tag{3.1}$$

For our purposes it is sufficient to consider the order ϵ^2 in the expansion (3.1); we also remark that we have neglected an error of the order $\mathrm{O}(|\zeta|^{N+1})$ in the normal form expansion so that our analysis will be valid in a region where this error is small with respect to ϵ^2. From the explicit form of the map M_ϵ, we obtain a relation between the expansion of the interpolating Hamiltonian and the expansion of the map. On the one hand, the Lie transformation $\exp(\epsilon\xi_n D_{V(J,\Theta)})$ explicitly reads

$$\begin{aligned} J_{n+1} &= J_n + \epsilon\xi_n[J, V](J_n,\Theta_n) + \tfrac{1}{2}\epsilon^2\xi_n^2[[J,V],V](J_n,\Theta_n) + \mathrm{O}(\epsilon^3) \\ \Theta_{n+1} &= \Theta_n + \epsilon\xi_n[\Theta, V](J_n,\Theta_n) + \tfrac{1}{2}\epsilon^2\xi_n^2[[\Theta,V],V](J_n,\Theta_n) + \mathrm{O}(\epsilon^3). \end{aligned} \tag{3.2}$$

On the other hand, the phase flow from the time 0 to the time 1 of the Hamiltonian (3.1), can be written in form (2.8) either by direct computation or by performing the change of variables defined by the time-dependent generating function $F = \hat{J}\Theta + H_0(\hat{J})(1-t)$ according to (the generic case from $t = n$ to $t = n+1$ can be analysed in the same way)

$$\left.\begin{aligned} \hat{J} &= J \\ \hat{\Theta} &= \Theta + \Omega(J)(1-t) \end{aligned}\right\} \qquad t \in [0, 1[. \tag{3.3}$$

In the new variables the Hamiltonian reads

$$\hat{H}(\hat{J},\hat{\Theta},t) = \epsilon\Xi(t)H_1(\hat{J}, \hat{\Theta} - \Omega(\hat{J})(1-t)) + \epsilon^2\Xi^2(t)H_2(\hat{J}, \hat{\Theta} - \Omega(\hat{J})(1-t))) + \mathrm{O}(\epsilon^3) \tag{3.4}$$

and the expansion of the phase flow is given by

$$\hat{J}_1 = \hat{J}_0 + \epsilon\xi_0 \int_0^1 [\hat{J}_0, H_1(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-t))]\,\mathrm{d}t + \epsilon^2\xi_0^2 \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s[[\hat{J}_0, H_1(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-t))], H_1(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-s))] + \epsilon^2\xi_0^2 \int_0^1 [\hat{J}_0, H_2(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-t))]\,\mathrm{d}t + \mathrm{O}(\epsilon^3)$$

$$\hat{\Theta}_1 = \hat{\Theta}_0 + \epsilon\xi_0 \int_0^1 [\hat{\Theta}_0, H_1(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-t))]\,\mathrm{d}t + \epsilon^2\xi_0^2 \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s[[\hat{\Theta}_0, H_1(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-t))], H_1(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-s))] + \epsilon^2\xi_0^2 \int_0^1 [\hat{\Theta}_0, H_2(\hat{J}_0, \hat{\Theta}_0 - \Omega(J_0)(1-t))]\,\mathrm{d}t + \mathrm{O}(\epsilon^3). \tag{3.5}$$

By comparing the expansions (3.2) and (3.5), we obtain the relations

$$\int_0^1 H_1(J, \Theta - \Omega(J)(1-t))\,\mathrm{d}t = V(J, \Theta)$$
$$\int_0^1 H_2(J, \Theta - \Omega(J)(1-t))\,\mathrm{d}t = \frac{1}{2}\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s[H_1(J, \Theta - \Omega(1-s)), H_1(J, \Theta - \Omega(1-t))] \tag{3.6}$$

which allow one to compute explicitly H_1 and H_2 from V. For the sake of simplicity we only report the formula for H_1. Letting $v_k(J)$ and $h_{1,k}(J)$, $k \in \mathbb{Z}$, be the k Fourier coefficients of V and of H_1, respectively, the following relation holds

$$h_{1,k}(J) = v_k(J)\frac{(k\Omega(J)/2)}{\sin(k\Omega(J)/2)}\mathrm{e}^{\mathrm{i}k\Omega(J)/2} \qquad k \neq 0 \text{ and } k\Omega(J) \neq 0 \bmod 2\pi \tag{3.7}$$

which shows the appearance of divisors; the condition $k\Omega(J) \neq 0 \bmod 2\pi$ amounts to demanding that the phase space region we are analysing is free from resonances of order k. Indeed we have a natural cut-off on the Fourier expansion due to the fact that we have performed the normal form reduction up to a finite order N (equation (2.3)) and the region where the normalizing transformation is defined cannot contain resonances of order less than N according to the normal forms theory [12]. Without loss of generality we suppose $v_0(J) = 0$ and we set $h_{1,0}(J) = 0$.

4. The diffusion equation

By using the interpolating Hamiltonian we derive the diffusion equation for the distribution in the action variable. One has to remark that the interpolating Hamiltonian is not uniquely defined but depends, for instance, on the choice of the stochastic process $\Xi(t)$. This freedom has not to change the statistical properties of the phase flow, so that we expect that the diffusion equation depends only on quantities which enter in the definition of the stochastic map (2.8).

We observe that $\Xi(t)$ is not a stationary process since the correlations are

$$\langle \Xi(t)\Xi(t+\tau)\rangle = \begin{cases} \sigma^2 & \text{if } [t] - t \leqslant \tau < [t] - t + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

By following the same procedure as [10], we start from the Liouville equation for the particle density $\rho(J, \Theta, t)$ which reads

$$\frac{\partial \rho}{\partial t} + \Omega \frac{\partial \rho}{\partial \Theta} + \epsilon \Xi[\rho, H_1] + \epsilon^2 \Xi^2[\rho, H_2] = 0 \tag{4.2}$$

and we separate the density into a mean and a fluctuating part

$$\rho = \rho_0 + \epsilon \rho_1 \qquad \rho_0 = \langle \rho \rangle \qquad \langle \rho_1 \rangle = 0. \tag{4.3}$$

Taking the mean value of equation (4.2) we obtain

$$\frac{\partial \rho_0}{\partial t} + \Omega \frac{\partial \rho_0}{\partial \Theta} + \epsilon^2 [\langle \Xi \rho_1 \rangle, H_1] + \epsilon^2 \sigma^2 [\rho_0, H_2] = 0. \tag{4.4}$$

Subtracting the equation (4.4) from (4.2), we have

$$\frac{\partial \rho_1}{\partial t} + \Omega \frac{\partial \rho_1}{\partial \Theta} = \Xi(t)[H_1, \rho_0] + \mathrm{O}(\epsilon) \tag{4.5}$$

whose solution can be explicitly given in the form

$$\rho_1(J, \Theta, t) = \int_{-t}^{0} \Xi(t+\tau)[H_1, \rho_0](J, \Theta + \Omega\tau, t+\tau)\, \mathrm{d}\tau + \mathrm{O}(\epsilon). \tag{4.6}$$

The Fokker–Planck equation for the average distribution ρ_0, is obtained by replacing (4.6) in equation (4.4), taking into account that the expectation value in equation (4.4) can be explicitly calculated according to

$$\begin{aligned}[\langle \Xi \rho_1 \rangle, H_1] &= \int_{-t}^{0} [\langle \Xi(t) \Xi(t+\tau) \rangle [H_1, \rho_0](J, \Theta + \Omega\tau, t+\tau), H_1(J, \Theta)]\, \mathrm{d}\tau \\ &= \sigma^2 \int_{[t]-t}^{0} [[H_1, \rho_0](J, \Theta + \Omega\tau, t+\tau), H_1(J, \Theta)]\, \mathrm{d}\tau \end{aligned} \tag{4.7}$$

where we have used equations (4.1). Instead of solving the full equation, we observe that the angle variable can be considered a fast variable in the diffusion limit $\epsilon \to 0$, $n \to \infty$, $\epsilon^2 n =$ constant. Indeed from the unperturbed dynamics

$$\Theta_{n+1} = \Theta_n + \Omega(J_n) \tag{4.8}$$

if J_n is a stochastic process with mean value J_0 and correlation $\langle \Delta J_n \Delta J_k \rangle \propto \epsilon^2 \min(n, k)$ (where $\Delta J_n = (J_n - J_0)$), then the following estimate holds,

$$\begin{aligned}\langle (\Delta\Theta_n)^2 \rangle &= \sum_{k,h=0}^{n-1} \langle (\Omega(J_k) - \Omega(J_0))(\Omega(J_h) - \Omega(J_0)) \rangle \\ &= \sum_{k,h=0}^{n-1} \left(\frac{\mathrm{d}\Omega}{\mathrm{d}J}(J_0) \right)^2 \langle \Delta J_k \Delta J_h \rangle \propto \left(\frac{\mathrm{d}\Omega}{\mathrm{d}J}(J_0) \right)^2 \\ &\quad \times \epsilon^2 \sum_{k,h=0}^{n-1} \min(h, k) \propto \left(\frac{\mathrm{d}\Omega}{\mathrm{d}J}(J_0) \right)^2 \epsilon^2 n^3 \end{aligned} \tag{4.9}$$

up to terms $\mathrm{O}(n^4 \epsilon^4)$. Equation (4.9) means that the angle variable is completely relaxed to the uniform distribution in the diffusion limit, the relaxation time being proportional to $\epsilon^{-2/3}$. A rigorous proof of this result for anisochronous Hamiltonian systems can be found in [13]. Since the action and angle time scales are separated, we consider the distribution $\rho(J, t)$, reached for $t \gg \epsilon^{-2/3}$ once the angle has relaxed; it will satisfy an equation where the differential operator is replaced by its angle average. The correct mathematical procedure consists in applying the averaging theorem: the distribution function $\rho(J, \Theta, t)$ is replaced by its angle average

$$\rho_0(J, t) = \frac{1}{2\pi} \int_0^{2\pi} \rho_0(J, \Theta, t)\, \mathrm{d}\Theta \tag{4.10}$$

and the same average is applied to the differential operator. Denoting with $\langle\,\rangle_\Theta$ the average on the angle variable Θ, we evaluate $\langle[\langle \Xi\rho_1\rangle, H_1]\rangle_\Theta$ using (4.7):

$$\begin{aligned}\langle[\langle \Xi\rho_1\rangle, H_1]\rangle_\Theta &= \int_{[t]-t}^{0} \langle[[H_1(J, \Theta + \Omega\tau), \rho_0(J, t)], H_1(J, \Theta)]\rangle_\Theta\, \mathrm{d}\tau \\ &= \int_{[t]-t}^{0} \left\langle \frac{\partial}{\partial\Theta} \left(\frac{\partial H_1}{\partial J}(J, \Theta) \frac{\partial H_1}{\partial \Theta}(J, \Theta + \Omega\tau)\right)\right\rangle_\Theta \mathrm{d}\tau \frac{\partial\rho_0}{\partial J}(J, t) \\ &\quad - \frac{\partial}{\partial J}\left(\int_{[t]-t}^{0} \left\langle \frac{\partial H_1}{\partial\Theta}(J, \Theta) \frac{\partial H_1}{\partial \Theta}(J, \Theta + \Omega\tau)\right\rangle_\Theta \mathrm{d}\tau \frac{\partial\rho_0}{\partial J}(J, t)\right).\end{aligned} \tag{4.11}$$

The first term on the right-hand side of equation (4.11) is manifestly zero. The second one can be simplified by observing that the distribution function has a variation of order ϵ^2 for $t \in [n, n+1[$, where n is any integer. As a consequence we average on t in any such interval by neglecting the variation of $\rho_0(J, t)$, namely for any $t \in [n, n+1[$ we replace the coefficient of $\partial\rho_0/\partial J$ with an integral in t over that interval. Observing that

$$\int_n^{n+1} \mathrm{d}t \int_{[t]-t}^{0} \mathrm{d}\tau\, f(\tau) = \int_{-1}^{0} \mathrm{d}s \int_s^0 \mathrm{d}\tau\, f(\tau) \tag{4.12}$$

we replace (4.11) with the following time-averaged expression in order to obtain a diffusion equation which does not depend on our choices in the construction of the interpolating Hamiltonian (3.1),

$$\begin{aligned}\langle[\langle \Xi\rho_1\rangle, H_1]\rangle_\Theta &= -\frac{\partial}{\partial J}\left(\int_{-1}^{0} \mathrm{d}s \int_s^0 \mathrm{d}\tau \left\langle \frac{\partial H_1}{\partial\Theta}(J, \Theta) \frac{\partial H_1}{\partial \Theta}(J, \Theta + \Omega\tau)\right\rangle_\Theta \frac{\partial\rho_0}{\partial J}(J, t)\right) \\ &= -\frac{1}{2}\frac{\partial}{\partial J}\left(\int_{-1}^{0} \mathrm{d}s \int_{-1}^0 \mathrm{d}\tau \left\langle \frac{\partial H_1}{\partial\Theta}(J, \Theta) \frac{\partial H_1}{\partial \Theta}(J, \Theta + \Omega(\tau - s))\right\rangle_\Theta \frac{\partial\rho_0}{\partial J}(J, t)\right)\end{aligned} \tag{4.13}$$

where we have used the following relation valid for every even function:

$$\int_{-1}^{0} \mathrm{d}s \int_s^0 \mathrm{d}\tau\, f(\tau) = \frac{1}{2} \int_{-1}^{0} \mathrm{d}s \int_{-1}^0 \mathrm{d}\tau\, f(\tau - s) \qquad f(\tau) = f(-\tau). \tag{4.14}$$

Using the invariance of the angle average by a translation ($\Theta \to \Theta + \Omega s$) and the first equation (3.6), which defines H_1, to replace it with V we obtain for $\langle[\Xi\rho_1\rangle, H_1]\rangle_\Theta$

$$\begin{aligned}&-\frac{1}{2}\frac{\partial}{\partial J}\left(\int_{-1}^{0} \mathrm{d}s \int_{-1}^0 \mathrm{d}\tau \left\langle \frac{\partial H_1}{\partial\Theta}(J, \Theta + \Omega s) \frac{\partial H_1}{\partial \Theta}(J, \Theta + \Omega\tau)\right\rangle_\Theta \frac{\partial\rho_0}{\partial J}(J, t)\right) \\ &\qquad = -\frac{1}{2}\frac{\partial}{\partial J}\left(\left\langle \left(\frac{\partial V}{\partial\Theta}(J, \Theta)\right)^2 \right\rangle_\Theta \frac{\partial\rho_0}{\partial J}(J, t)\right).\end{aligned} \tag{4.15}$$

Observing that

$$\langle[\rho_0(J, t), H_2]\rangle_\Theta = 0 \tag{4.16}$$

and defining the diffusion coefficient by

$$D(J) = \frac{\epsilon^2\sigma^2}{2} \left\langle \left(\frac{\partial V}{\partial\Theta}(J, \Theta)\right)^2 \right\rangle_\Theta \tag{4.17}$$

the Fokker–Planck equation for the distribution $\rho_0(J,t)$ reads

$$\frac{\partial \rho_0}{\partial t}(J,t) = \frac{\partial}{\partial J} D(J) \frac{\partial \rho_0}{\partial J}(J,t). \tag{4.18}$$

It is straightforward to observe that the diffusion coefficient $D(J)$ depends only on the map (3.2) and not on our choices in computing the interpolating Hamiltonian.

The proposed approach can be generalized to the case of multidimensional symplectic maps which can be reduced to the form (2.8). We only report the final result for the diffusion equation in the action variables J_i, $i = 1, \ldots, d$:

$$\frac{\partial \rho_0}{\partial t}(\boldsymbol{J},t) = \sum_{i=1}^{d}\sum_{k=1}^{d} \frac{\partial}{\partial J_i} D_{i,k}(\boldsymbol{J}) \frac{\partial}{\partial J_k} \rho_0(\boldsymbol{J},t) \tag{4.19}$$

where

$$D_{i,k}(\boldsymbol{J}) = \frac{\epsilon^2\sigma^2}{2}\left\langle \frac{\partial V}{\partial \Theta_i}\frac{\partial V}{\partial \Theta_k}\right\rangle_{\boldsymbol{\Theta}} \tag{4.20}$$

and the symbol $\langle\,\rangle_{\boldsymbol{\Theta}}$ means the average over all the angle variables Θ_i, $i = 1, \ldots, d$.

5. Numerical results

In order to compare the solution of the diffusion equation (4.18) and the distribution function calculated by iterating the initial map (2.1), we consider the well known Hènon map M perturbed by a random rotation matrix

$$\begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = R(\epsilon\xi_n)R(\omega)\begin{pmatrix} x_n \\ p_n + x_n^2 \end{pmatrix}. \tag{5.1}$$

We have chosen $\omega = 2\pi(\sqrt{5}-1)/2$, so that we have no macroscopic low-order nonlinear resonance in the unperturbed phase space (see figure 1(a)). By using a terminology of accelerator physics, we call the dynamical aperture A of an area preserving map M

$$A = \sup\{r / \|M^{\circ k}(r,0)\| < \infty \ \forall k \in \mathbb{Z}\}. \tag{5.2}$$

For our choice of ω the dynamical aperture is approximately 0.55; by calculating the normal form U up to order 10 according to equation (2.3), we have a good agreement between the normal form dynamics generated by $\Phi \circ U \circ \Phi^{-1}$ and the initial dynamics up to 80% of the dynamical aperture (see figure 1(b)): the distance of the orbits in the outer region is less than 1%. The diffusion coefficient (4.17) is a polynomial in $\sqrt{J}$, so that we cannot solve analytically the diffusion equation (4.18) and we have used a numerical integrator to compute the distribution function.

In order to simulate the effect of the dynamical aperture in the diffusion equation (4.18), we have inserted an absorbing boundary condition at the action which corresponds to the orbit through the point $(0.5, 0.0)$; this choice is justified by the fact that the diffusion coefficient (4.17) is divergent at the dynamical aperture (see also [11]). We have considered a noise ξ_n as a sequence of independent random variables uniformly distributed between $[-1, 1]$, so that the man value is zero and the second moment (see equation (4.1)) is $\sigma^2 = \frac{1}{3}$.

In figure 2(a) we compare the distribution functions computed by the diffusion equation (4.18) by using a numerical algorithm (continuous curve) and the direct iteration of population in the phase space by using the map (5.1) (histograms); the integration of the diffusion equation (4.18) is much faster (more than a factor 10^3) than the simulations with the map (5.1). We have made the simulations starting from a Gaussian distribution in the action, whose mean value corresponds to the orbit through the point $(0.25, 0.0)$, whereas

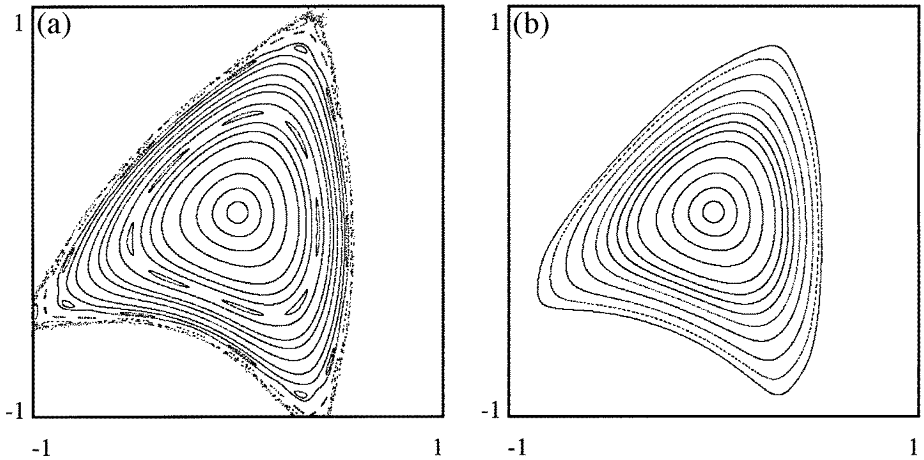

Figure 1. Phase space of the Hènon map computed by (a) directly iterating the map and (b) using the normal form at order 10. The frequency $\omega/2\pi$ is equal to the golden mean.

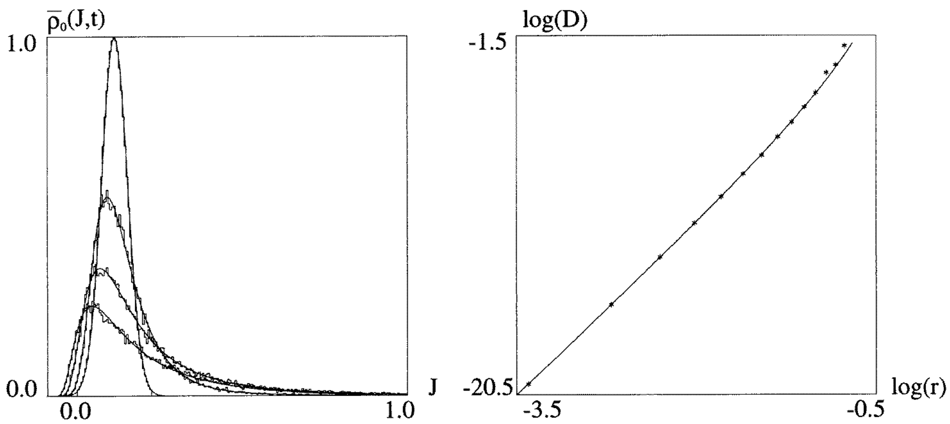

Figure 2. (a) Comparison of the distribution functions for the action variables computed by a direct simulation (histograms) and by using the Fokker–Planck equation (continuous curves). (b) Comparison of the diffusion coefficients computed by using the normal form at order 10 (continuous curve) and the direct calculation (stars) according to equation (5.3).

the initial phases are randomly distributed in $[0, 2\pi]$; the value of ϵ was fixed at $0.005 \times 2\pi$ and we have used a population of 35 000 particles for the simulation. The different curves (in decreasing order), refer to different iteration numbers $n = 0, 2000, 6000, 12\,000$. The agreement between the histograms and the continuous curves is very good.

In figure 2(b) we compare the logarithm of the diffusion coefficient (4.17) computed by using the normal forms at order 10 (continuous curve) with the same quantity computed by means of the numerical simulations (stars) according to the equation

$$D(J_0) = \lim_{n\to\infty} \frac{1}{2n} \langle (J_n - J_0)^2 \rangle \qquad \epsilon^2 n \ll 1 \tag{5.3}$$

in the x-axis instead of the action J, we report the logarithm of the abscissa r of the points $(r, 0)$, which belong to the corresponding unperturbed orbits. The numerical values are obtained with $\epsilon = 0.001 \times 2\pi$, $\sigma^2 = \frac{1}{3}$ and $n = 100$; we see that we have a good agreement between the two diffusion coefficients up to 80% of the dynamical aperture.

6. Conclusions

The proposed method to derive a diffusion equation for the action distribution function of an area-preserving map stochastically perturbed seems to be very promising to describe the diffusive phenomena observed in physical experiments when a weak noise together with nonlinear effects is present. In particular, this method could be applied to the symplectic maps which describe the betatronic motion in a particle accelerator in order to simulate the effect of a noise in the feeding current of the quadrupoles.

Appendix

In this appendix we derive equations (3.5) and (3.6), which define the interpolating Hamiltonian for the map (2.8). By integrating the canonical equations associated to the Hamiltonian (3.4), we obtain for $t \in [0, 1[$

$$\begin{aligned}\hat{J}(t) &= \hat{J}_0 + \epsilon\xi_0 \int_0^t [\hat{J}(\tau), H_1(\tau)]\,\mathrm{d}\tau + \epsilon^2\xi_0 \int_0^t [\hat{J}(\tau), H_2(\tau)]\,\mathrm{d}\tau \\ \hat{\Theta}(t) &= \Theta_0 + \epsilon\xi_0 \int_0^t [\hat{\Theta}(\tau), H_1(\tau)]\,\mathrm{d}\tau + \epsilon^2\xi_0 \int_0^t [\hat{\Theta}(\tau), H_2(\tau)]\,\mathrm{d}\tau \end{aligned} \tag{A.1}$$

where for the sake of notation $H_j(\tau)$ means $H_j(\hat{J}, \hat{\Theta} - \Omega(\hat{J})(1-\tau))$, $j = 1, 2$. Therefore expansions (3.5) follow by substituting recursively $\hat{J}(\tau)$ and $\hat{\Theta}(\tau)$ with the expansions in the right-hand side of equation (A.1) up to terms of order ϵ^2. By a direct comparison of the expansions (3.2) and (3.5), the first equation (3.6) follows immediately. In order to derive the second equation, which defines H_2, we substitute the function $V(J, \Theta)$ in equations (3.2) according to the first equation (3.6), so that the comparison of the second-order terms provides

$$\begin{aligned}\int_0^1 \mathrm{d}t[\hat{J}_0, H_2(t)] &= -\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s[[\hat{J}_0, H_1(t)], H_1(s)] + \frac{1}{2}\int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s[[\hat{J}_0, H_1(t)], H_1(s)] \\ &= \frac{1}{2}\int_0^1 \mathrm{d}t \int_t^1 \mathrm{d}s[[\hat{J}_0, H_1(t)], H_1(s)] - \frac{1}{2}\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s[[\hat{J}_0, H_1(t)], H_1(s)] \\ &= \frac{1}{2}\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s[[H_1(t), H_1(s)], \hat{J}_0] \end{aligned} \tag{A.2}$$

where we have used the Jacobi identity for the Poisson bracket to compute the second term in the second equation (A.2). An analogous expression can be obtained for $[\hat{\Theta}_0, H_2(t)]$ so that the second equation (3.6) holds.

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